CS201: DISCRETE COMPUTATIONAL STRUCTURES **Semester III**

Module II

Syllabus: *Review of Permutations and combinations, Principle of inclusion exclusion, Pigeon Hole Principle,*

Recurrence Relations: *Introduction- Linear recurrence relations with constant coefficients - Homogeneous solutions - Particular solutions - Total solutions*

Algebraic systems:- *Semigroups and monoids - Homomorphism, Subsemigroups and submonoids*

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2.1 Review of Permutations and combinations

Definition 2.1 *If we have a ways of doing something and b ways of doing another thing and we can not do both at the same time, then there are $a + b$ ways to choose one of the actions.*

Example:

A woman has decided to shop at one store today, either in the north part of town or the south part of town. If she visits the north part of town, she will shop at either a mall, a furniture store, or a jewellery store (3 ways). If she visits the south part of town then she will shop at either a clothing store or a shoe store (2 ways).

Thus there are $3 + 2 = 5$ possible shops the woman could end up shopping at today.

Definition 2.2 *If there are a ways of doing something and b ways of doing another thing, then there are axb ways of performing both actions.*

Example:

A drama club is holding audition for a play. Six men and Eight women came for auditioning. The director can cast the **leading** couple in $6 * 8 = 48$ ways.

2.1.1 Permutations

Definition 2.3 *Given a collection of n objects, the arrangements of these objects is called a permutation. If there are n objects and number of permutation of size r is given by $\frac{n!}{(n-r)!}$*

Example:

In a class of 10 students, 5 are to be chosen and seated in a row for a picture. How many such arrangements are possible?

Of the 5 position, any of the 10 can take the first position. For the second position any of the remaining 9 can take. Continuing this way fifth position can be occupied from a possibility of 6 students. Then the number of arrangements is given by

$10 * 9 * 8 * 7 * 6$

By the definition of permutation, the number of permutations of size 5 for the 10 students is given by $\frac{10!}{(10-5)!}$

2.1.2 Combinations

Definition 2.4 *Number of possible combinations of r objects from a set of n objects*

$$\frac{n!}{r!(n-r)!}$$

When order is relevant we think in terms of permutations. When order is not relevant combinations could be used.

2.2 Recurrence Relation

Recurrence relations are those functions which depends on some of the prior terms. A numeric function can be described by a recurrence relation together with an appropriate set of boundary conditions. The numeric function is also referred to as the solution of the recurrence relation.

2.2.1 Linear recurrence relations with Constant Coefficients

A recurrence relation of the form $c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + c_3 a_{r-3} + \ldots + c_k a_{r-k} = f(r)$. where c_i's are constants, is called a linear recurrence relation with constant coeffidents. Above recurrence relation is known as a k^{th}-order recurrence relation, provided that both C_0 and C_k are nonzero.

2.2.2 Homogeneous solutions

The (total) solution of a linear difference equation with constant coefficients is the sum of two parts, the homogeneous solution, which satisfies the difference equation when the right-hand side of the equation is set to 0, and the particular solution, which satisfies the difference equation with $f(r)$ on the right-hand side. The discrete numeric function that is the solution of the difference equation is the sum of two discrete numeric functions - one is the **homogeneous solution** and the other is the **particular solution**.

Let $a^{(h)} = (a_0^{(h)}, a_1^{(h)}, \ldots, a_r^{(h)}, \ldots)$ denote the homogeneous solution and $a^{(p)} = (a_0^{(p)}, a_1^{(p)}, \ldots, a_r^{(p)}, \ldots)$ denote the particular solution to the difference equation.

$$c_0 a_r^{(h)} + c_1 a_{r-1}^{(h)} + c_2 a_{r-2}^{(h)} + c_3 a_{r-2}^{(h)} + \ldots + c_k a_{r-k}^{(h)} = 0$$

and

$$c_0 a_r^{(p)} + c_1 a_{r-1}^{(p)} + c_2 a_{r-2}^{(p)} + c_3 a_{r-2}^{(p)} + \ldots + c_k a_{r-k}^{(p)} = f(r)$$

Total solution is

$$a = a^{(h)} + a^{(p)}$$

A homogeneous solution of a linear difference equation with constant coefficients is of the form $A\alpha_1^r$ where α_1 is called a *characteristic root* and A is a constant determined by the boundary conditions. Substituting $A\alpha^r$ for a_r in the difference equation with the right-hand side of the equation set to 0, we obtain

$$c_0 A\alpha^r + c_1 A\alpha^{r-1} + c_2 A\alpha^{r-2} + c_3 A\alpha^{r-3} + \ldots + c_k A\alpha^{r-k} = 0$$

which can be simplified to

$$c_0 \alpha^r + c_1 \alpha^{r-1} + c_2 \alpha^{r-2} + c_3 \alpha^{r-3} + \ldots + c_k \alpha^{r-k} = 0$$

The above equation is called the characteristic equation of the difference equation. Therefore, if $\alpha_!$ is one of the roots of the characteristic equation (it is for this reason that a! is called a characteristic root), $A\alpha_1^r$ is a homogeneous solution to the difference equation. A characteristic equation of k^{th} degree has K characteristic roots.

Example: The recurrence relation for the Fibonacci sequence is

$$a_r = a_{r-1} + a_{r-2}$$

The corresponding characteristic equation is

We have

$$a_r = a_{r-1} + a_{r-2}$$

$$a^r = a^{r-1} + a^{r-2} \tag{2.1}$$
$$\alpha^r = \alpha^{r-1} + \alpha^{r-2} \tag{2.2}$$
$$\alpha^{r-(r-2)} = \alpha^{(r-1)-(r-2)} + \alpha^{(r-2)-(r-2)} \tag{2.3}$$
$$\alpha^2 = \alpha^1 + \alpha^0 \tag{2.4}$$
$$\alpha^2 - \alpha - 1 = 0 \tag{2.5}$$
(2.6)

This has two solutions

$\alpha_1 = \frac{1+\sqrt{5}}{2} \quad \alpha_2 = \frac{1-\sqrt{5}}{2}$

Therefore the homogeneous solution is

$$a_r^{(h)} = A_1\left(\frac{1+\sqrt{5}}{2}\right)^r + A_2\left(\frac{1-\sqrt{5}}{2}\right)^r$$

Two constants A_1 and A_2 are to be determined from the boundary conditions $a_0 = 1$ and $a_1 = 1$.

2.2.3 Particular solutions

There is no general procedure for determining the particular solution of a difference equation. However, in simple cases, this solution can be obtained by the method of inspection.

case 1: If RHS $f(r) = \beta$, where β is a constant , then take particular Solution as $a_r^{(p)} = P$. Where P is a constant to be determined.

Example: Consider the relation

$$a_r - 5a_{r-1} + 6a_{r-2} = 1$$

Since $f(r)$ is a constant , the particular solution will also be a constant P. Substituting P in LHS

$P - 5P + 6P = 1$

$P = 1/2$

$ar^{(p)} = 1/2$

case 2: When $f(r)$ is of the form of a polynomial of degree t in r

$$F_1r^t + F_2r^{t-1} + \ldots + F_tr + F_{t+1}$$

The corresponding particular solution will be of the form

$$P_1r^t + P_2r^{t-1} + \ldots + P_tr + P_{t+1}$$

Example: Consider the relation

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$$

We assume that the general form of the particular solution is

$$P_1r^2 + P_2r + P_3$$

Where P1 , P2 and P3 are constants to be determined . Substituting the expression into LHS we obtain,

$$P_1r^2 + P_2r + P_3 + 5P_1(r-1)^2 + 5P_2(r-1) + 5P_3 + 6P_1(r-2)^2 + 6P_2(r-2) + 6P_3$$

which simplifies to

$$12P_1r^2 - (34P_1 - 12P_2)r + (29P_1 - 17P_2 + 12p_3)$$

comparing with RHS

$12P_1 = 3$

$34P_1 - 12P_2 = 0$

$29P_1 - 17P_2 + 12P_3 = 0$

$P_1 = 1/4; P_2 = 17/24; P_3 = 115/288$

So particular solution is

$$a_r^{(p)=}(1/4)r^2 + (17/24)r + (115/288)$$

case 3: When $f(r)$ is of the form β^r , then corresponding particular solution is of the form $P\beta^r$, if β is not a characteristic root of the recurrence relation.

example : Consider the recurrence relation

$$a_r + a_{r-1} = 3r2^r$$

The general form of the particular solution is

$$(P_1 r + P_2)2^r$$

substituting into LHS we obtain,

$$(P_1 r + P_2)2^r + [P_1(r-1) + P_2]2^{r-1} = 3r2^r$$

Comparing both sides ,

$$(3/2)P_1 = 3$$

$$(-1/2)P_1 + (3/2)P_2 = 0$$

$$P_1 = 2$$

$$P_2 = 2/3$$

$$a_r^{(p)} = (2^r + 2/3)2^r$$

case 4:

When $f(r)$ is of the form β^r , and β is a characteristic root of multiplicity $m-1$, then corresponding particular solution is of the form $r^{m-1}(P_1 r^t + P_2 r^{t-1} + \ldots + P_t r + P_{t+1})\beta r$

Example: Find particular solution of the difference equation $a_r - 2a_{r-1} = 3.2^r$

Assume general form of the solution is

$$r^{m-1}(P_1 r^2 + P_2 r + P_3)\beta^r$$

, where P_1, P_2, P_3 are constants to be determined.

Substituting the assumed solution to the given equation we get

$$Pr2^r$$

(Because 2 is a characteristic root of multiplicity 1)

$$Pr2^r - 2P(r-1)2^{r-1} = 3.2^r$$

$$P.2^r = 3.2^r$$

$$P = 3$$

Thus particular solution is

$$a_r^{(p)} = 3r2^r$$

2.2.4 Total Solutions

We must combine the homogeneous solution and the particular solution and determine the undetermined coefficients in the homogeneous solution.

Example : Consider the recurrence relation

$$a_r - 5a_{r-1} + 6a_{r-2} - 5 = 1$$

Characteristic equation is

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\alpha_1 = 3$$

$$\alpha_2 = 2$$

Thus homogeneous solution

$$a_r(h) = A_1 3^r + A_2 2^r$$

Since f(r) is a constant ,

$$a_r(p) = P$$

$$P - 5P + 6P = 1$$

$$P = 1/2$$

Total solution = Homogeneous solution + Particular solution

$$a_r = A_1 3^r + A_2 2^r + 1/2$$

2.3 Algebraic Systems

Definition 2.5 *A system consisting of a non-empty set and one or more n-ary operations on the set is called an Algebraic System. An Algebraic System will be denoted by $S, f1, f2....$, when S is the non-empty set and $f1, f2...$ are n-ary operations on S.*

General Properties of Algebraic Systems

Let $S, *, \oplus$ be an algebraic system ,where $*$ and $\oplus$ are binary operations on S. (Not necessarily usual addition and multiplication).

Closure Property

For any $a, b \in S, a * b \in S$.

Associativity

For any $a, b \in S, (a * b) * c = a * (b * c)$

Commutativity

for any $a, b \in S, a * b = b * a$

Identity Element

There exists a distinguished element $e \in S$, such that for any $a \in S$, $a * e = e * a = a$

Inverse Element

For each $a \in S$, there exists an element $a^{-1} \in S$ such that $a * a^{-1} = a^{-1} * a = e$.

The element $a^{-1} \in S$ is called the inverse of a under operation $*$.

Distributivity

for any $a, b, c \in S, a * (b \oplus c) = a * b \oplus a * c$ In this case $*$ is said to be distributive over $\oplus$

Example :

The algebraic system $(z, +, .)$ with usual addition and multiplication satisfies all properties. $(z, +, .)$ satisfies Associative property for $+$ and . , Commutative property for $+$ and . , and Distributive law . Identity element is 0 for addition and 1 for multiplication. For each element , there exists an element negative of a, also called inverse element.

2.3.1 Semigroups and Monoids

Definition 2.6 *A **semigroup** is a non-empty set S together with an associative binary operation $*$ defined on S. We shall denote the semigroup by $(S, *)$, (when it is clear what the operation $*$ is, simply by S)*

*If * is commutative then the semigroup is said to be **commutative or abelian semigroup.***

Definition 2.7 *A **monoid** is a semigroup $(S, *)$ that has an identity ϕ.*

Example 1: Let z^+ be the set of positive integers $\{1, 2, 3, 4...\}$. Then $(z^+, +)$ is a semigroup with usual binary operation of addition. However $(z^+, +)$ is not a monoid since there is no additive identity. Note that $(z^+, +)$ is a commutative semigroup.

Example 2:

Let N ={0,1,2,3...} be the set of natural numbers.Then $(N, +)$ is a commutative monoid with identity $e = 1$**.** Obviously $(N, +1)$ is an abelian semigroup.

2.3.2 Homomorphism

Definition 2.8 *Let $(S, *)$ and (T, Δ) be any two semigroup. A function $f : S \longrightarrow T$ is called **semigroup homomorphism** if for any two elements $a, b \in s$ we have*

$f(a * b) = f(a)\Delta f(b)$

If f is one -to-one , onto or one-to-one onto then the semigroup homomorphism is known as semigroup monomorphism , epimorphism or isomorphism, respectively.If there is a semigroup isomorphism from S onto T then the two semigroup $(S, *)$ and (T, Δ) are said to be isomorphic.

2.3.3 Subsemigroups and Submonoids

Definition 2.9 *Let (S,*) be a semigroup and let T be a subset of S. If T is closed under the operation * (that is , a*b $\in$ T , whenever a and b are elements of T) then (T,*) is called a **subsemigroup** of (S,*).*

Similarly let (S,) be a monoid with identity e $\in$ T , and let T be a non-empty subset of S. If T is closed under operation * and e$\in$ T , then (T,*) is called a **submonoid** of (S,*).*

Example 1:

For the semigroup $(N,+), (z^+,+)$ is a subsemigroup since z^+ the set of positive integers $\{1,2,3...\}$ is a subset of N and Z^+ is closed under operation $+$.

Example 2:

For the semigroup $(N,+), (T,+)$ where T is the set of odd integers {1,3,5...} is not a subsemigroup since T is not closed under the binary operation $+$. (sum of two odd numbers is even)

Example 3:

For the monoid $(R,.,1)$ the set of real numbers ,$(N,.,1)$ the set of natural numbers is a submonoid since N subset of R, N is closed under . and identity $1 \in N$, whereas $(E,.)$ where E is set of even positive integers is not a submonoid since identity $1 \notin E$ although E subset of R , and E is closed under .