CS201: DISCRETE COMPUTATIONAL STRUCTURES **Semester III**

Module IV

Syllabus: ***Lattices and Boolean Algebra***: *Lattices - Sublattices - Complete lattices - Bounded Lattices - Complemented Lattices - Distributive Lattices - Lattice Homomorphisms.*

Boolean algebra – sub algebra, direct product and homomorphisms

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Federal Institute of Science And Technology (FISAT)

Contents

5.1 Lattices

Definition 5.1 *A **lattice** is a poset (L,≤) in which every subset {a,b} consisting of two elements has a least upper bound and a greatest lower bound.*

Least Upper Bound (LUB) of $(\{a, b\})$ is denoted by $a \vee b$ and call its as join of *a* and *b*. Greatest Lower Bound (GLB) of $(\{a, b\})$ is denoted by $a \wedge b$ and call its as meet of *a* and *b*.

Example: Let S be a set and $L = P(S)$. $\subseteq$, containment is a partial order on L. Then $a \vee b$ is the set $A \cup B$ and $a \wedge b$ is the set $A \cap B$.

Theorem 5.1 $(P(S), \subseteq)$ *is a Lattice*

Proof: We know that $\subseteq$ is a poset on $P(S)$. Then
$X \subseteq T, Y \subseteq T$ means $X \cup Y \subseteq T$; and
$X \subseteq T, Y \subseteq T$ means $T \subseteq X \cap Y$
So $X \cap Y$ is the infimum and $X \cup Y$ is the supremum of $\{X, Y\}$.
Hence $(P(S), \subseteq)$ is a Lattice. ∎

5.1.1 Sublattices

Definition 5.2 *Let $(L, \leq)$ be a lattice. A nonempty subset S of L is called a sublattice of L if $a \vee b \in S$ and $a \wedge b \in S$ whenever $a \in S$ and $b \in S$*

5.1.2 Complete lattices

Definition 5.3 *A poset $(L, \leq)$ is called a complete lattice if every subset M of L has a least upper bound (supremum) and a greatest lower bound (infimum) in $(L, \leq)$.*

5.1.3 Bounded Lattices

Definition 5.4 *A lattice L is said to be bounded if it has a greatest element I and a least element O*

For example the lattice Z^+ under partial order of divisibility is not a bounded lattice since it has no greatest element.

The lattice $P(S)$ of all subsets of a set S, is bounded. Its greatest element is S and its least element is ϕ.

5.1.4 Complemented Lattices

Definition 5.5 *Let L be a bounded lattice with greatest element I and least element O, and let $a \in L$. An element $a' \in L$ is called a* ***complement*** *of a if*

$$a \vee a' = I \text{ and } a \wedge a' = O$$

Observe that

$$O' = I \text{ and } I' = O$$

In general an element may have more than one complement.

Definition 5.6 *A* ***complemented lattice*** *is a bounded lattice in which every element a has a complement.*

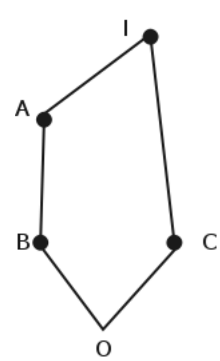

Figure 5.1: Complemented Lattice

The figure represents a complemented lattice since every element a has a complement

Element	Complement
I	O
A	C
B	C
C	{B,A}
O	I

5.1.5 Distributive Lattices

Definition 5.7 *A lattice L is called distributive if for any elements a, b and c in L we have the following distributive properties*

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

5.1.6 Lattice Homomorphisms

Definition 5.8 *Let L and M be lattices. A map ϕ from L to M is called a lattice homomorphism if ϕ respects meet and join. That is, for $a, b \in L$*

1. $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$
2. $\phi(a \vee b) = \phi(a) \vee \phi(b)$

5.2 Boolean algebra

Definition 5.9 *Let L and K be lattices, and let $\phi : L \to K$. A lattice isomorphism is a one-to-one and onto lattice homomorphism.*

If the Hasse daigram of the lattice corresponding to a set with n elements is labeled by sequences of 0's and 1's of length n; then the resulting lattice is named B_n

Definition 5.10 *A finite lattice is called a Boolean algebra if its is isomorphic with B_n for some nonnegative integer n.*

5.2.1 Subalgebra

Definition 5.11 *Let A be a Boolean algebra and B a non-empty subset of A. Consider the following conditions*

1. *if $a \in B$, then $a' \in B$*
2. *if $a, b \in B$, then $(a \vee b) \in B$*
3. *if $a, b \in B$, then $(a \wedge b) \in B$*

A non-empty subset B of a Boolean algebra A satisfying conditions 1 and 2 (or equivalently 1 and 3) is called a Boolean subalgebra of A.

5.2.2 Direct product

Direct product of an algebraic object is given by the Cartesian product of its elements, considered as sets.

Definition 5.12 *Let* $(B_1, \vee_1, \wedge_1,', O_1, I_1)$ *and* $(B_2, \vee_2, \wedge_2,'', O_2, I_2)$ *be two boolean algebras. The direct product of the two boolean algebras is defined to be a boolean algebra that is given by* $(B_1 X B_2, \vee_3, \wedge_3,''', O_3, I_3)$ *in which the operations are defined for any* (a_1, b_1) *and* $(a_2, b_2) \in B_1 X B_2$ *as*

1. $(a_1, b_1) \vee_3 (a_2, b_2) = (a_1 \vee_1 a_2, b_1 \vee_2 b_2)$
2. $(a_1, b_1) \wedge_3 (a_2, b_2) = (a_1 \wedge_1 a_2, b_1 \wedge_2 b_2)$
3. $(a_1, b_1)''' = (a_1', b_1'')$
4. $O_3 = (O_1, O_2)$ *and* $I_3 = (I_1, I_2)$

5.2.3 Homomorphisms

Definition 5.13 *Let* $(B_1, \vee_1, \wedge_1,', O_1, I_1)$ *and* $(B_2, \vee_2, \wedge_2,'', O_2, I_2)$ *be two boolean algebras. A mapping* $f : B_1 \to B_2$ *is called a Boolean homomorphism if all the operations of the Boolean algebra are preserved; i.e., for any* $a, b \in B$

1. $f(a \vee_1 b) = f(a) \vee_2 f(b)$
2. $f(a \wedge_1 b) = f(a) \wedge_2 f(b)$
3. $f(a') = f(a)''$
4. $f(O_1) = O_2$
5. $f(I_1) = I_2$