CS201: DISCRETE COMPUTATIONAL STRUCTURES **Semester III**

Module V

Syllabus: ***Propositional Logic:***- *Propositions - Logical connectives - Truth tables, Tautologies and contradictions - Contra positive - Logical equivalences and implications DeMorgan's Laws - Rules of inference : Validity of arguments.*

Disclaimer: *These may be distributed outside this class only with the permission of the Instructor.*

Contents

1.1 Propositional Logic

Definition 1.1 *Propositional logic (also called Propositional calculus) is the branch of mathematical logic concerned with the study of propositions (whether they are true or false) that are formed by other propositions with the use of logical connectives.*

1.1.1 Propositions

Definition 1.2 *A proposition is a statement or assertion that expresses a judgement or opinion.*

Sentences considered in propositional logic are not arbitrary sentences but are the ones that are either true or false, but not both. For example, "Sky is blue", and "2 + 5 = 9" are propositions, with the second one being a false proposition.

Sentences like "Open the door", and "Is it raining outside ?" are not propositions. "x + 5 = 9 is also not a proposition because it is not true or false. Its truth or falsity depends on the value of x, and so it is called a **predicate**.

In logic a statement means any declarative sentence (including mathematical ones such as equations) that is true or false or could become true or false in the presence of additional information.

Definition 1.3 *A statement is called atomic if it has no connectives or quantifiers. Those statements which contain one or more primary stements and some connectives are called compound statements. They are also called as* ***statement formulas***

1.1.2 Logical connectives

Logical connective are symbols or words used to connect two or more sentences. The main connectives are

1. Not: $\neg$
2. Conjunction(And): $\wedge$
3. Disjunction(Or): $\vee$
4. Implies: $\rightarrow$
5. If and only if: $\leftrightarrow$

1.1.2.1 Logical Not

The negation of a statement is formed by by introducing "not" at proper place in a statement. If *P* denotes a statement, then the negation of "P" is written as "¬P" and read as "not P". For example the statement

P: Kochi is a city

Then ¬P is

¬P: Kochi is not a city

P	$\neg P$
T	F
F	T

Table 1.1: Truth Table

1.1.2.2 Conjunction

Conjunction of two statements P and Q is the statement $P \wedge Q$ which is read as "P and Q". $P \wedge Q$ has the truth value T whenever both P and Q have the truth value T; otherwise it has the truth value F. For example the statement

P: It is raining today Q: There are 61 students in this class

$P \wedge Q$: It is raining today **and** there are 61 students in this class. "And" is sometimes used in a different sense so it cannot be used in place of $\wedge$. For example the sentence *"He opened the book and started to read"*

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.2: Truth Table

1.1.2.3 Disjunction

Conjunction of two statements P and Q is the statement $P \vee Q$ which is read as "P or Q". $P \vee Q$ has the truth value F whenever both P and Q have the truth value F; otherwise it has the truth value T. Connective $\vee$ is not always same as the word "or". For example the sentence

I shall watch the game on television or go to the game.

In the above sentence "or" is used in the exclusive sense; that is one or the other possibility exists but not both.

The symbol $\vee$ comes from the Latin word "*vel*" which means "inlcusive or". An example is the sentence

There is something wrong with the bulb or with the wiring

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.3: Truth Table

1.1.2.4 Implies (Conditional)

If P and Q are two statements, then the statement $P \rightarrow Q$ which is read as "If P, then Q" is called a conditional statement. This statement has a truth value F when Q has truth value F and P has truth value T; otherwise it has truth value T. The statement P is called antecedent and Q is called consequent.

In ordinary speech, a statement of the form "P implies Q" normally asserts that there is some causal relationship between P and Q, that the truth of P somehow causes the truth of Q. This is not required in mathematics or logic. Also it can be noted that P and Q are not symmetric in $P \rightarrow Q$; meaning $P \rightarrow Q$ does not mean that $Q \rightarrow P$.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.4: Truth Table

1.1.2.5 If and only if

Equivalence of two formulas can be represented with $\leftrightarrow$. If A and B are equivalent we can write $A \leftrightarrow B$. $P \leftrightarrow Q$ is true if and only if P and Q are both true or both false.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 1.5: Truth Table

1.1.3 Tautologies

A well formed formula (wff) or statement formula can be generated by

1. A statement variable standing alone
2. If A is a wff, then $\neg A$ is a wff
3. If A and B are wff; then $A \vee B$, $A \wedge B$, $A \rightarrow B$, $A \leftrightarrow B$ are wff
4. A string of symbols containing the statement variables, connectives, and parenthesis is a wff, if it can be obtained by finitely many applications of the rules 1,2,3

Definition 1.4 *A statement formula which is true regardless of the truth values of the statements is called a* ***tautology*** *or* ***logical truth*** *or* ***universally valid formula****.*

A statement formula which is false regardless of the truth values of the statements is called a ***contradiction****.*

For example, $P \vee \neg P$ is True independent of the statement P. So the statement formula $P \vee \neg P$ is a tautology. The statement formula $P \wedge \neg P$ is a contradiction.

1.1.4 Logical equivalences

Definition 1.5 *If truth values of two statement formulas A and B are equal for every one of the 2^n possible sets of truth values assigned to variables occurring in A and B; then A and B are said to be* ***equivalent****.*

For example $P \vee P$ is equivalent to P

P	$P \vee P$
T	T
F	F

Table 1.6: Truth Table

$(P \wedge \neg P) \vee Q$ is equivalent to Q

P	**Q**	$(P \wedge \neg P) \vee Q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**F**

Table 1.7: Truth Table

- Equivalence is a symmetric relation
- Equivalence is a transitive relation

Equivalence of two formulas A and B is written as $A \iff B$

1.1.4.1 Equivalent formulas

- Idempotent Laws
  - $P \vee P \iff P$
  - $P \wedge P \iff P$
- Associative Laws
  - $(P \vee Q) \vee R \iff P \vee (Q \vee R)$
  - $(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$
- Commutative Laws
  - $P \vee Q \iff Q \vee P$
  - $P \wedge Q \iff Q \wedge P$
- Distributive Laws
  - $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
  - $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
- Absorption Laws
  - $P \vee (P \wedge Q) \iff P$
  - $P \wedge (P \vee Q) \iff P$
- De Morgan's Laws
  - $\neg(P \vee Q) \iff \neg P \wedge \neg Q$
  - $\neg(P \wedge Q) \iff \neg P \vee \neg Q$

1.1.4.2 Tautological Implication

A statement A is said to tautologically imply a statement B if and only if $A \rightarrow B$ is a tautology. This is denoted by $A \implies B$. For example $P \wedge Q \implies P$

P	Q	$P \wedge Q$	$P \wedge Q \rightarrow P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Table 1.8: Truth Table

Since $P \wedge Q \rightarrow P$ is a tautology, we can say that $P \wedge Q \implies P$.

1.1.5 Rules of inference

This refers to inferring a conclusion from certain premises. The process of conclusion is called as deduction or formal proof. In formal proof every step at any stage is acknowledged. The conclusions, known as theorems can be inferred from a set of premises called axioms.

Rules of inference are criteria for determining the validity of an argument. Rules will be given in the form of statement formulas. Any conclusion which is arrived at by using rules of inferences is called a valid conclusion. And the argument is called a valid argument.

1.1.5.1 Validity of Arguments

If A and B be two statement formulas; we say "*B is a valid conclusion of the premise A*", iff $A \to B$ is a tautology.

We say that, from a set of premises $\{H_1, H_2, \ldots, H_n\}$ a conclusion C follows logically iff

$H_1 \land H_2 \land \ldots \land H_n \implies C$

For determining whether a conclusion C follows can be found by constructing a truth table. For example

$H_1 : P \to Q$
$H_2 : P$
$C : Q$

$H_2 : P$	$C : Q$	$H_1 : P \to Q$	$H_1 \land H_2$	$H_1 \land H_2 \to C$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Table 1.9: Truth Table

Since $H_1 \land H_2 \to C$ is a tautology, we can say that $H_1 \land H_2 \implies C$. This means that C follows from the premises H_1 and H_2